ME 243

Mechanics of Solids

Lecture 4: Torsion and helical springs

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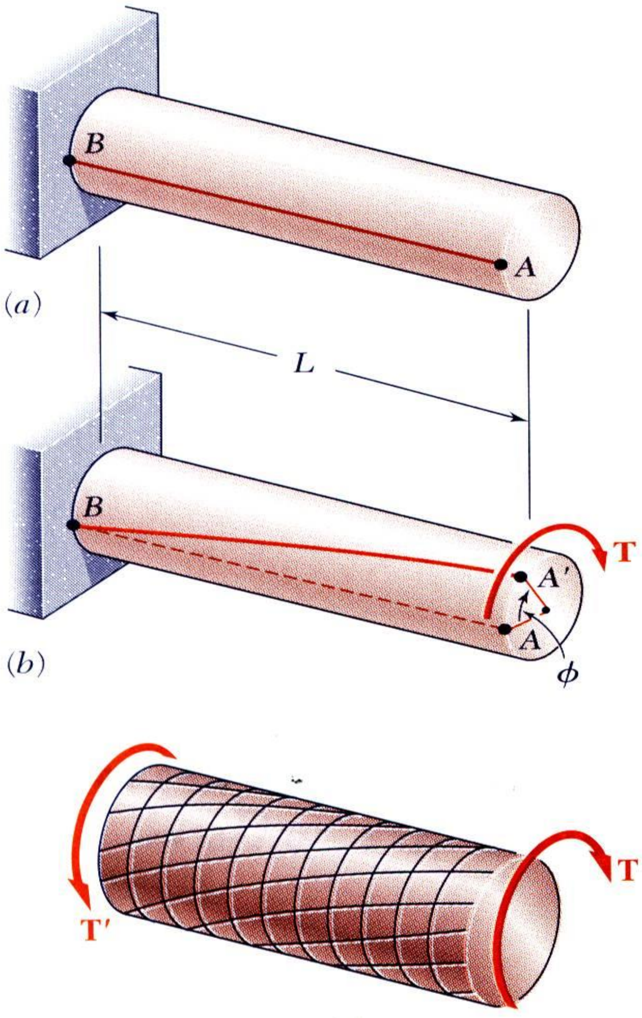
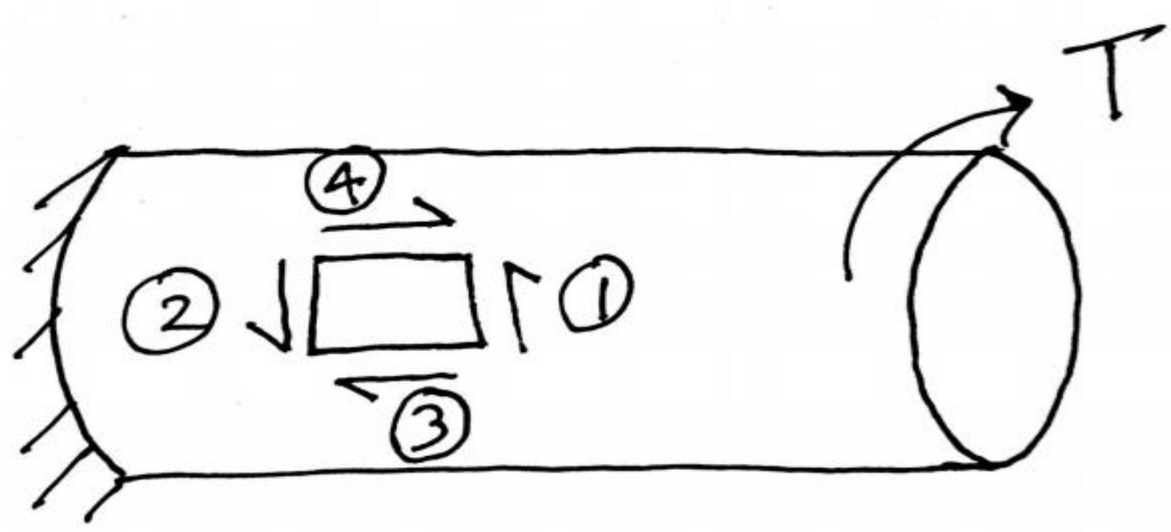
Torsion



- When one end of a circular member is kept fixed and a twisting couple is applied at the other end, an angular deformation of free end is found with respect to fixed end. The state of the member is called torsion.

- Shear stress is developed in this type of loading.

Torsion



From free body diagram of a fiber on the shaft surface,

1. Applied force
2. Reaction force

3 & 4 : Induced force

Assumptions

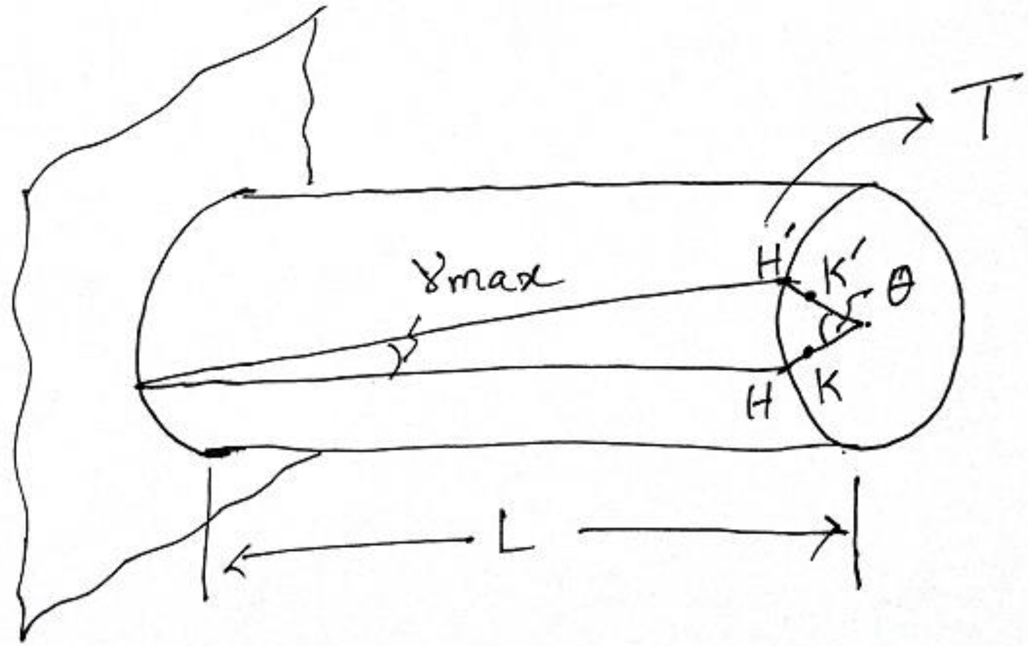
- The material is homogeneous.
- Circular sections remain circular.
- Plane sections remain plane and do not warp.
- The shearing strain varies linearly with the distance from the central axis of the circular member.
- Shaft is loaded by twisting couples in planes that are perpendicular to the axis of the shaft.
- Stresses do not excess the proportional limit.

Torsional shear stress

γ_{max}= shear strain at outer radius (c)

γ= shear strain at radius 'r'

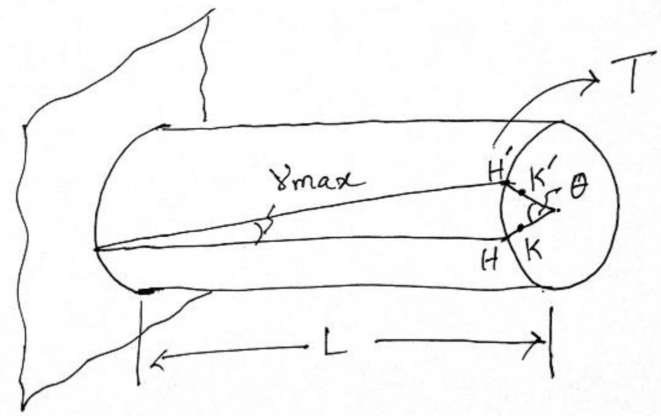
θ= Angle of twist



$$\tan \gamma_{max} \approx \gamma_{max} = \frac{HH'}{L} = \frac{c\theta}{L} \quad \ldots (1)$$

Again $$\tan \gamma' \approx \gamma' = \frac{KK'}{L} = \frac{r\theta}{L} \quad \ldots (2)$$

Torsional shear stress



From equation 1 and 2,

$$\therefore \frac{\gamma'}{\gamma_{max}} = \frac{r}{c},$$

But, $\tau = \gamma G$

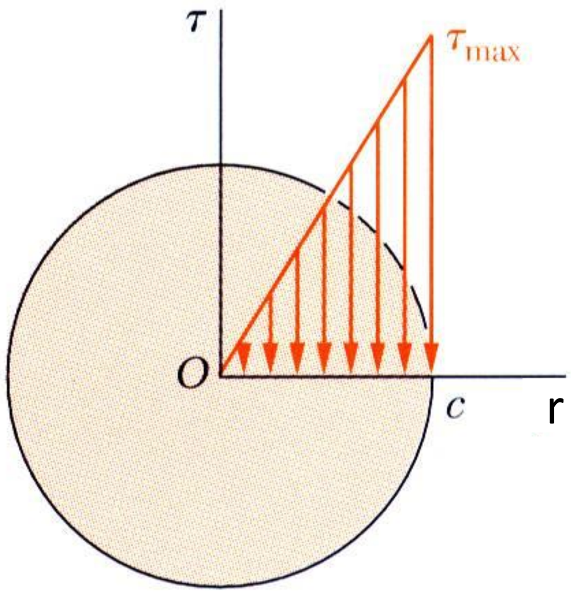
Where, τ = Torsional shear stress

G= Shear modulus of rigidity

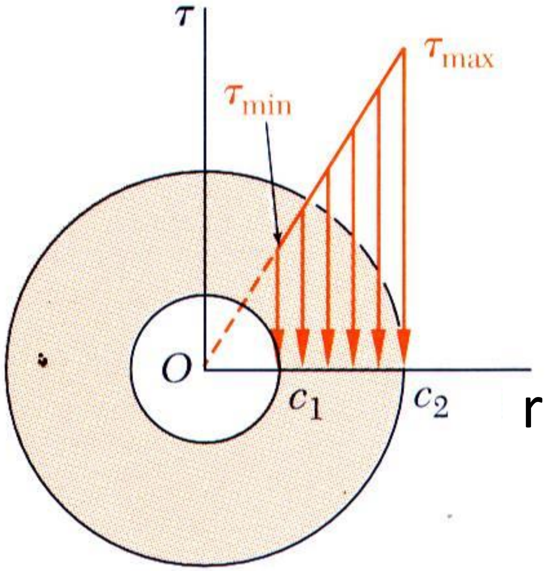
Therefore,

$$\frac{\tau}{\tau_{max}} = \frac{r}{c} \quad \text{or} \quad \boxed{\tau = \frac{r}{c}\tau_{max}}$$

Torsional shear stress distribution throughout the radius of shaft



Solid shaft

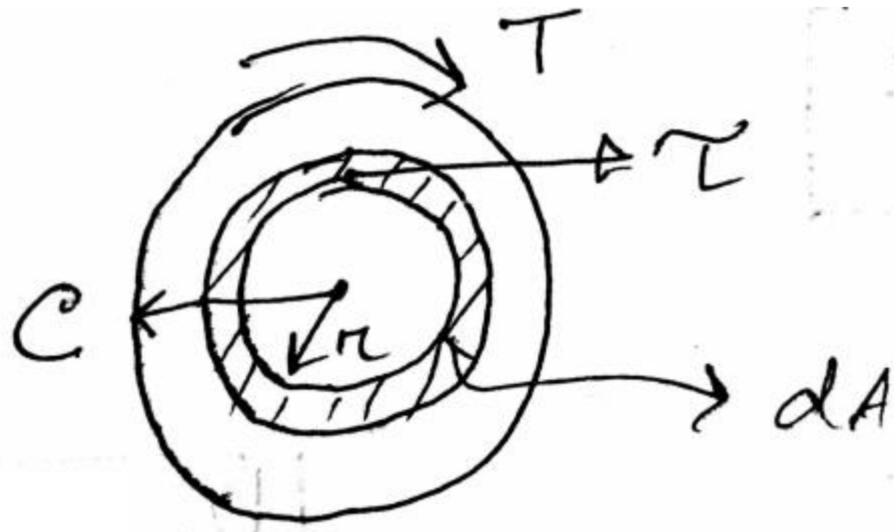


Hollow shaft

- At the center of the shaft, torsional shear stress is zero

Torsion formula

Let us choose a differential area 'dA' at a radius 'r' from the axis of shaft. Let the shear stress at radius r is τ.



Now , at differential area 'dA'

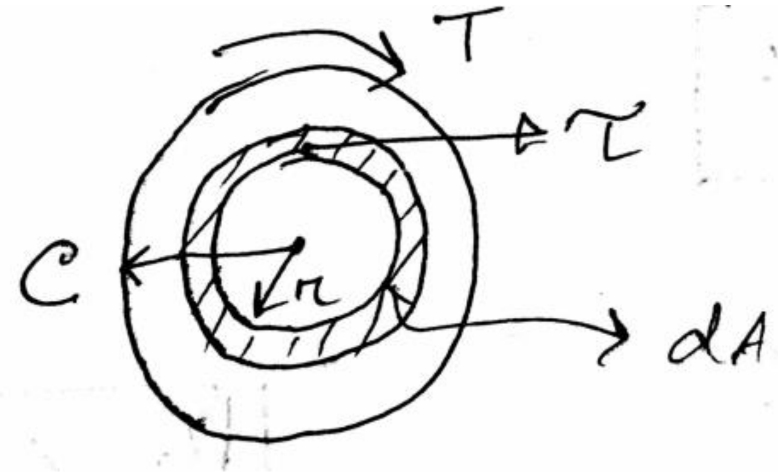
$$\text{Force} = \tau \times dA$$

$$\text{Moment} = r \cdot \text{Force}$$

$$= r \cdot \frac{r}{c} \tau_{max} \, dA$$

$$= \frac{\tau_{max}}{c} r^2 \, dA$$

Torsion formula



The sum of the resisting moments is equal to applied torque.

Therefore,

$$T = \int_0^c \frac{\tau_{max}}{c} r^2 \, dA$$

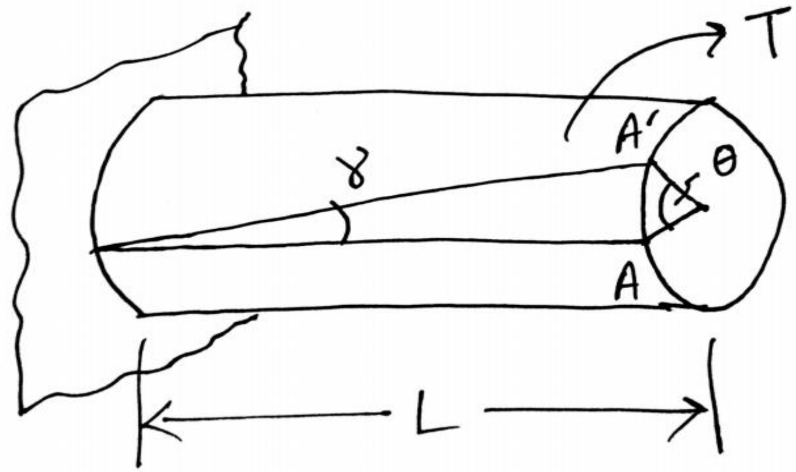
$$= \frac{\tau_{max}}{c} \int_0^c r^2 \, dA$$

,But, $\int_0^c r^2 \, dA$ = polar moment of inertia of the cross section of the bar, **J**

Therefore, $T = \frac{\tau_{max}}{c} J$

$$\therefore \tau_{max} = \frac{Tc}{J} \quad \text{or} \quad \tau = \frac{Tr}{J}$$

Angle of twist



$$\gamma = \frac{AA'}{L}$$

$$\therefore AA' = L\gamma = c\theta$$

$$\therefore \gamma = \gamma_{max} = \frac{c\theta}{L} \quad \cdots (1)$$

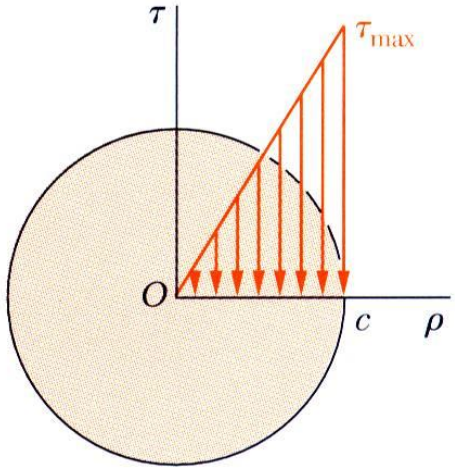
Again, $\tau_{max} = \gamma_{max} \cdot G$

$$\therefore \gamma_{max} = \frac{\tau_{max}}{G} = \frac{Tc}{JG} \quad \cdots (2)$$

$\therefore$ From (1) & (2),

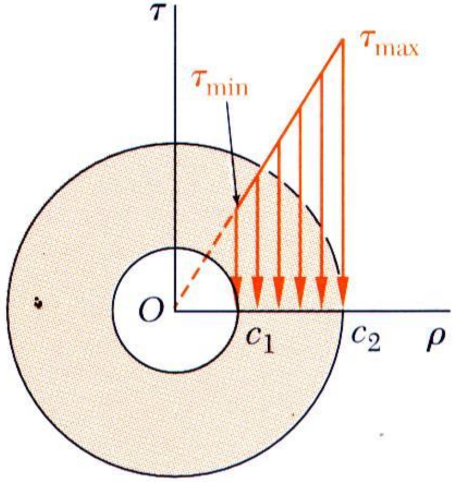
$$\boxed{\theta = \frac{TL}{GJ}}$$

Polar moment of inertia



For solid shaft,

$$J = \frac{1}{2}\pi c^4$$

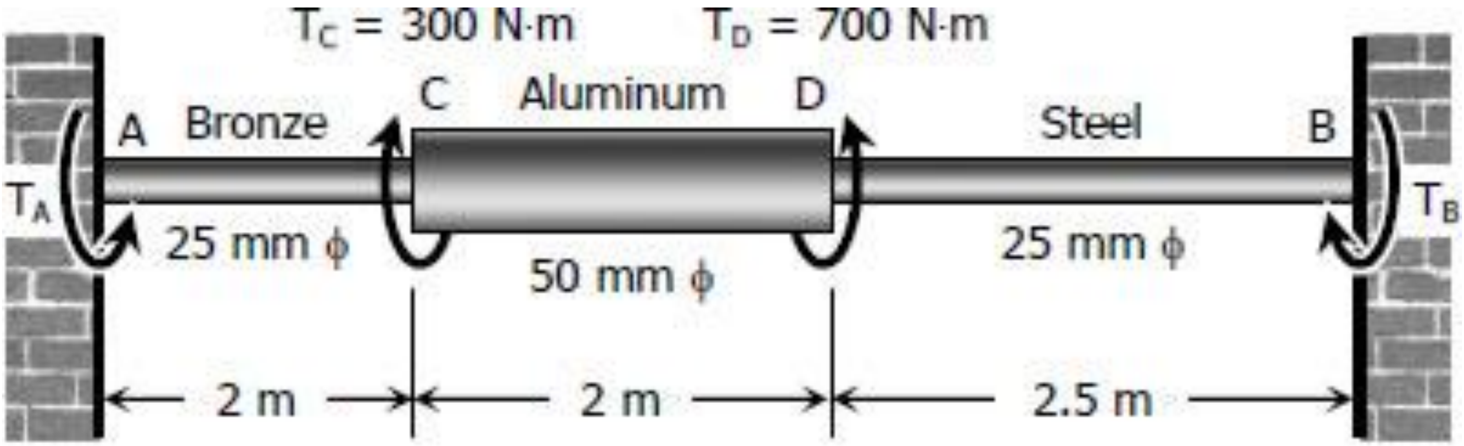


For hollow shaft,

$$J = \frac{1}{2}\pi\left(c_2^4 - c_1^4\right)$$

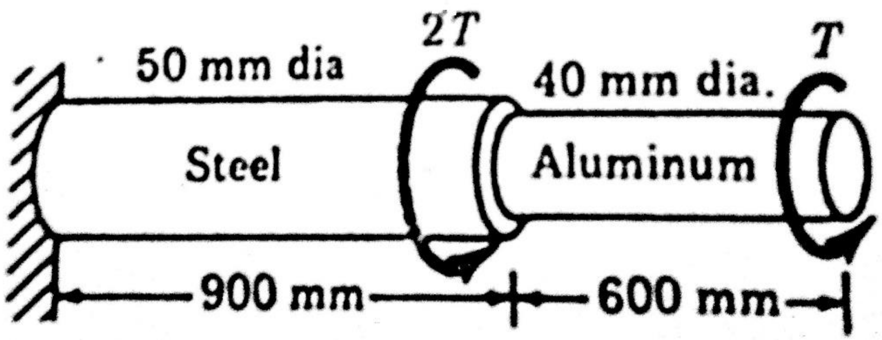
Problem# 323 (singer)

- A shaft composed of segments AC, CD, and DB is fastened to rigid supports and loaded as shown in Fig. For bronze, G = 35 GPa; aluminum, G = 28 GPa, and for steel, G = 83 GPa. Determine the maximum shearing stress developed in each segment.



Problem # 316 (singer)

- A compound shaft consisting of a steel segment and an aluminum segment is acted upon by two torques as shown in Fig. Determine the maximum permissible value of T subject to the following conditions: τ_{st}= 83 MPa, τ_{al}= 55 MPa, and the angle of rotation of the free end is limited to 6°. For steel, G = 83 GPa and for aluminum, G = 28 GPa.



Spring

- A spring is an elastic object used to store mechanical energy, subsequently release it, to absorb shock, or to maintain a force between contacting surfaces.
- Types of spring:

Based on force:

1. Extension spring
2. Compression spring
3. Torsional spring

Based on shape:

1. Coil shaped
2. Flat shaped

- Coil springs are of two types:

1. Helical spring
2. Conical spring

Types of spring

Extension spring

Flat Leaf spring

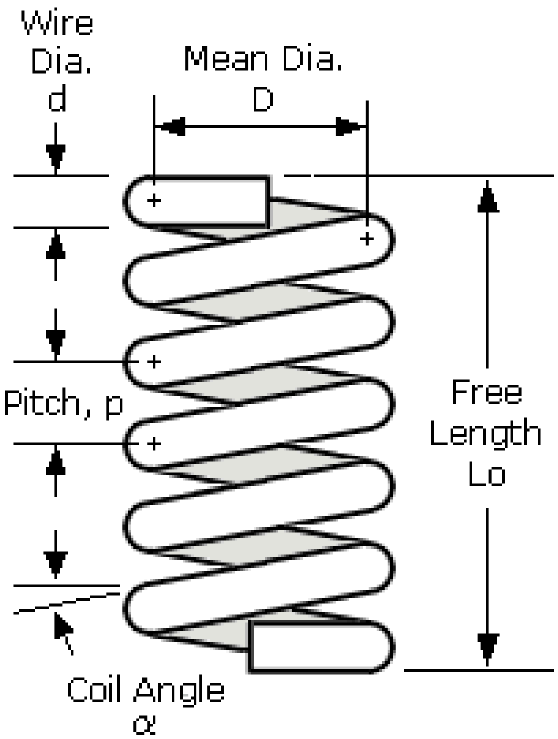
Torsional spring

Compression spring

Conical spring

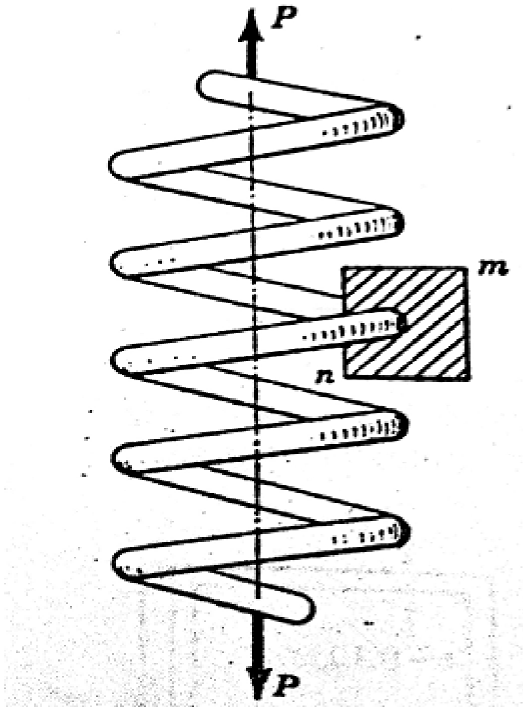
Helical spring

- The helical springs are made up of a wire coiled around a cylinder in the form of a helix.



Helical spring

Let us consider that helical spring is elongated by an axial force ‘P’. The wire diameter is ‘d’ and the mean radius of spring is ‘R’. The helix angle is small, so that any coil of the spring may be considered as lying approximately in the plane perpendicular to the axis of the spring.



Helical spring

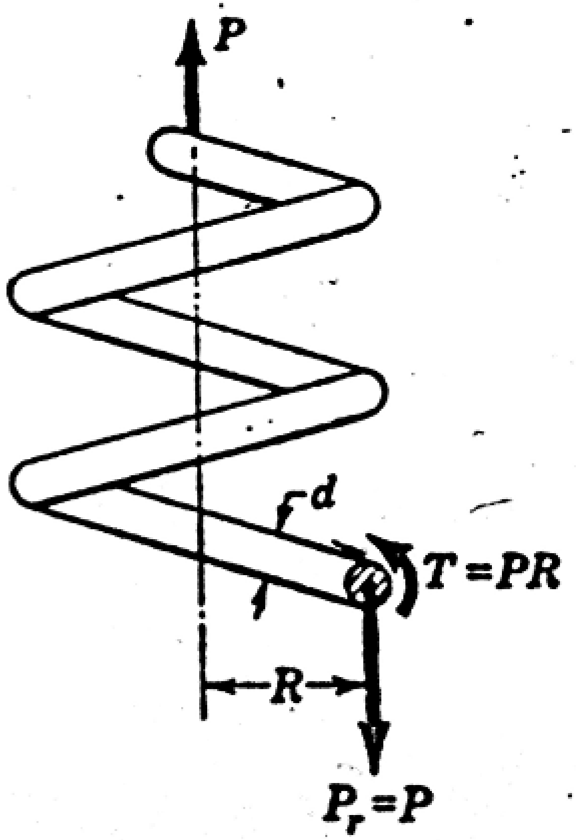
From the free body diagram of the upper half of the spring,

P_r= Resisting force

T = Resisting torque

So, two types of stresses are developed,

- Direct shear stress due to force P_r
- Torsional shear stress due to torque T



Helical spring

- The maximum shearing stress is the sum of the direct shearing stress τ_1 = P/A and the torsional shearing stress τ_2 = Tr/J, with T = PR.

$$\tau = \tau_1 + \tau_2$$

$$\tau = \frac{P}{\pi d^2/4} + \frac{16PR}{\pi d^3}$$

Therefore,

$$\tau = \frac{16PR}{\pi d^3}\left(1 + \frac{d}{4R}\right)$$

This formula neglects the curvature of the spring. This is used for light spring where the ratio d/4R is small.

Helical spring

- For heavy springs and considering the curvature of the spring, A.M. Wahl formula is more precise, it is given by:

$$\tau = \frac{16PR}{\pi d^3}\left(\frac{4m-1}{4m-4} + \frac{0.615}{m}\right)$$

Where, m=(2R/d)=(D/d) = spring index and

(4m - 1)/(4m - 4) is the Wahl Factor

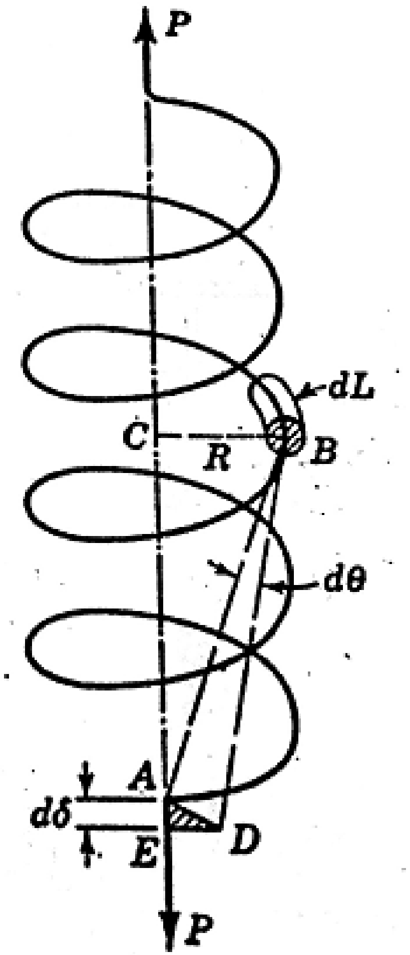
- Deflection due to direct shear stress is neglected compared to deflection for torsional shear.

Deflection of helical spring

Let us consider all of the spring is rigid except the small length dL.
After applying the load, the end A will rotate to D through the small angle dθ.

As dθ is small, the arc AD = AB. dθ may be considered as straight line perpendicular to AB.

So, from similar triangles ADE and BAC, we obtain,



$$\frac{AE}{AD} = \frac{BC}{AB}$$

Or, $$\frac{d\delta}{AB \cdot d\theta} = \frac{R}{AB}$$

Or, $$d\delta = R\, d\theta$$

Deflection of helical spring

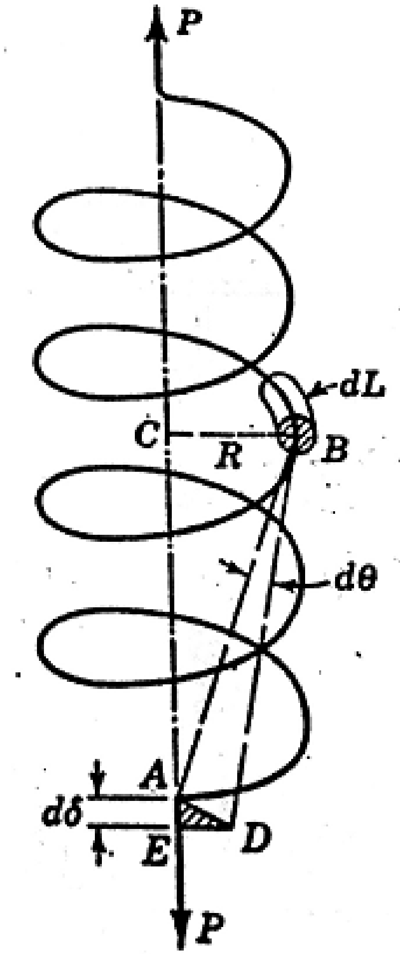
Replacing the value of dθ, we get,

$$d\delta = R\frac{(PR)\,dL}{JG}$$

Integrating, $\delta = \dfrac{PR^2L}{JG}$

Replacing L by 2πRn, which is the length of n coils of mean radius R and J by $\pi d^4/32$, we get,

$$\delta = \frac{64PR^3n}{Gd^4}$$



Spring constant of helical spring

- The deformation δ is directly proportional to the applied load P.
- The ratio of P to δ is called the spring constant or spring stiffness, k.

$$k = \frac{P}{\delta} = \frac{Gd^4}{64R^3n}$$

Problem# 349 (singer)

- A rigid bar, hinged at one end, is supported by two identical springs as shown in Fig. Each spring consists of 20 turns of 10-mm wire having a mean diameter of 150 mm. Compute the maximum shearing stress in the springs, Neglect the mass of the rigid bar.

Figure P-349